Robust passivity-based control of switched-reluctance motors

Antonio Loria, G. Espinosa-Pérez, Erik Chumacero

Robust passivity-based control of switched-reluctance motors

Antonio Loría Gerardo Espinosa–Pérez Erik Chumacero

Abstract

We propose a state-feedback controller for switched-reluctance motors as a preliminary step towards the solution of the sensorless control problem (without measurement of rotor variables). We establish global exponential stability. Furthermore, our controller renders the closed-loop system robust to external disturbances that is, input-to-state stable. Although there exist some works on sensorless control of switched-reluctance motors, these consist mainly on *ad hoc* solutions without theoretical foundation. The few theoretically-validated results in the literature are established under more stringent conditions such as knowledge of the load torque.

I. INTRODUCTION

In view of their many technological advantages such as their absence of permanent magnets or windings in the rotor, switched-reluctance machines are highly reliable and have lower costs than other synchronous motors. These make them particularly attractive in a number of applications such as transportation systems and the boosting industry of electrical vehicles –see [1], [2], [3], [4]. However, technological simplicity comes at the cost of model mathematical complexity: due to magnetic saturation, the map flux-current is highly nonlinear; also, the mechanical torque is a nonlinear function of the stator currents and angular positions. Accounting for a few exceptions –see [5], [6], magnetic saturation is commonly neglected in the dynamic model hence, it is assumed that the mapping from flux to current is linear (the inductance depends purely on rotor angles). Even under such simplification, the model of the switched-reluctance motor is a complex nonlinear multivariable system which posses significant challenges to the control theorist and the control practitioner.

In spite of a number of articles on control of switched-reluctance machines via full state feedback –[7], [5], [6], [8] and partial state-feedback –[9], [10], articles including a theoretical analysis are scarce. Certainly the same holds for *ad hoc* solutions based on methods such as model-predictive control —[11].

As for other electro-mechanical machines, a natural approach is to use two loops in the control. The first to drive the rotor variables (velocity and position) to a desired reference; as a matter of fact, typically only the velocities account as variables of interest to be controlled. A second loop is closed around the stator dynamics via current feedback; the goal is to steer the currents to a regime such that the current drives the rotor velocities to the desired reference. Although appealing, this method is obstructed by the fact that currents enter nonlinearly in the mechanics equations. That is, the rotor dynamics consists in a drift-less system non-affine in the control input. To overcome the difficulty of control implementation, the *torque sharing* technique is adopted –see [12], [5], [13]. It exploits the physical properties of the machine by 'allocating' the control action through one phase at a time.

In this paper we address the problem of velocity/position control of switched reluctance motors via the approach described above. We use full-state feedback that is, we assume that both velocities and positions as well as currents, are measured. It is also assumed that the load torque is unknown and constant. Although we use full-state measurement, we provide proofs of Lyapunov global exponential stability in closed loop. Also, we establish that the closed-loop system is globally input to state stable with respect to external additive disturbances. Therefore, the controller that we propose constitutes a first step towards control of reluctance drives under more realistic assumptions: partial state feedback, account of magnetic saturation, *etc.*

A. Loría is with CNRS. E. Chumacero is with Univ. Paris Sud. Address: LSS-SUPELEC, 91192 Gif–sur–Yvette, France. E-mail: antonio.loria@lss.supelec.fr

G. Espinosa–Pérez is with DEPFI – UNAM, A.P. 70-256, 04510 México D.F., MEXICO. gerardoe@servidor.unam.mx.

The rest of the paper is organized as follows. In the next section we present the dynamic model, we assume that the inductances are functions of the rotor angular positions only. For clarity of exposition, in Section IV we describe the first control loop: for the rotor dynamics; in Section V we present the stator-currents controller and in Section VI we present our main results.

II. The motor equations

A. Problem statement

Considering the experimentally-validated fact that the mutual inductance among stator phases is negligible, a general three-phase dynamic model is given by *–cf.* [14],

$$\dot{\psi}_j(\theta, x) + Rx_j = u_j, \qquad j = 1, 2, 3; \tag{1a}$$

$$J\dot{\omega} = \mathbf{T}_e(\theta, x) - T_L(\theta, \omega) \tag{1b}$$

$$\dot{\theta} = \omega. \tag{1c}$$

Equation (1a) corresponds to the stator dynamics and Equations (1b), (1c) describe the rotor's motion. For each phase j, u_j is the voltage applied to the stator terminals, ψ_j is the flux linkage and x_j is the stator current; $x = [x_1,\, x_2,\, x_3]^\top$. Based on the assumption that the machine operates at relatively low current levels, it is common practice to express the inductance of each phase as a strictly positive Fourier series truncated at the first harmonic that is, the flux is represented by a linear function of the currents: $\psi_j(\theta, x) = L_j(\theta)x_j$ where

$$L_j(\theta) = \ell_0 - \ell_1 \cos\left(N_r\theta - (j-1)\frac{2\pi}{3}\right)$$

with $\ell_0 > \ell_1 > 0$ hence, the stator dynamics equation becomes

$$u_j = L_j(\theta)\dot{x}_j + K_j(\theta)\omega x_j + Rx_j \tag{2}$$

where

$$K_j(\theta) = \frac{\partial L_j}{\partial \theta} = N_r\ell_1 \sin\left(N_r\theta - (j-1)\frac{2\pi}{3}\right)$$

corresponds to the phase-inductance variation relative to the rotor angular position.

In Equations (1b), (1c) R represents the stator resistance, J corresponds to the total rotor inertia, θ and ω denote the angular position and velocity respectively. The inputs are the mechanical torque of electrical origin, $\mathbf{T}_e$ and the load torque, T_L. Based on the assumption that inductances are decoupled, $\mathbf{T}_e$ corresponds to the sum of the torques produced by each phase *i.e.*,

$$\mathbf{T}_e = \frac{1}{2}\sum_{j=1}^{3} K_j(\theta)x_j^2.$$

Although the previously-described model is simplified for the purpose of control design and stability *analysis*, it is adopted in both the electrical-machines and the control research communities *–cf.* [14]. Other models as for instance that used in [9], account for variations of ℓ_0 and ℓ_1 depending on the stator currents but does not include theoretical validation the experimental results reported therein. A fully-nonlinear has been used in a few *experimental* works such as [15] and [5] however, the controllers proposed therein use full state-feedback.

The **control problem** consists in driving the angular velocity ω to a set-point reference ω^*. It is assumed that T_L is constant and unknown.

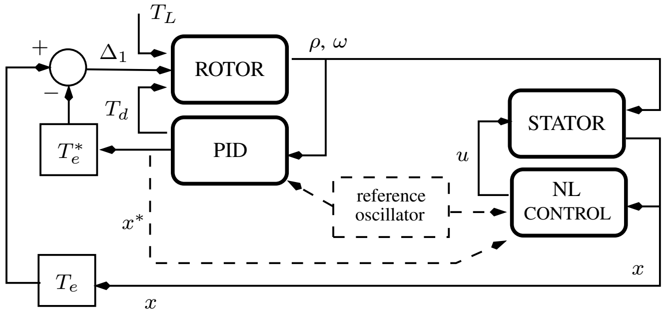

Fig. 1. Illustration of the control approach. A PID controller virtually injected through the variable T_d steers $\omega \to \omega^*$ –See Section IV. T_d is also injected in the form of a reference current x^* into the stator control loop and a nonlinear controller ensures current tracking control –see Section V. The systems are feedback interconnected through the nonlinear map T_e. The proper definition of the reference model (dashed lines) ensures that the interconnection remains stable in view of a small-gain argument –see Section VI.

III. THE CONTROL APPROACH

Generally speaking, an appealing control approach for electro-mechanical machines is to design a control law for the mechanical part (the rotor) separately from a controller for the electrical part (the stator). The control action on the rotor dynamics enters through the mechanical torque; naturally, the current x may be seen as a virtual control input in (1b). Accordingly, a control law u may be designed for the stator equations (1a) and implemented by applying the corresponding input voltage. The control u must be such that the actual current x tracks a desired reference x^* which is viewed as the control law for the rotor equations. See Figure 1.

However appealing, this approach is stymied by two major technical difficulties:

- the rotor equation (1b) is non-affine in the ‘control input’ x,
- θ appears non-linearly.

The first difficulty is addressed in Section III-B via the so-called *torque-sharing* approach, adapted to the purpose of this paper –*cf.* [12], [5], [13]. The second presents an obstacle to observer-design and output-feedback control. Although the controller that we present uses full-state feedback, it constitutes a first step towards sensorless control. The approach relies on a modified dynamic model, equivalent to (1b)-(1c), (2) and which is propitious to certainty-equivalence control; this is presented in Section III-A.

A. New coordinates

Following ideas from [16] we introduce the function $\varrho : [-\pi,\, \pi] \to \mathbb{S}^a$ where $\mathbb{S}^a := \{(\varrho_1,\, \varrho_2) \in \mathbb{R}^2 : \varrho_1^2 + \varrho_2^2 = a\}$, with $a \in \mathbb{R}_+$. Let $\vartheta \in [-\pi,\, \pi]$, $A > 0$ and

$$\varrho_1 \;:=\; A\cos\big(N_r[\theta+\vartheta]\big) \tag{3a}$$
$$\varrho_2 \;:=\; A\sin\big(N_r[\theta+\vartheta]\big). \tag{3b}$$

Now, if we set $\vartheta = -\theta(0)$ and for any $\rho_\circ \in \mathbb{R}_+$, $A := \rho_\circ$ we see that the solutions of

$$\dot\rho = \omega \mathbb{J}\rho, \quad \rho = [\rho_1, \rho_2]^\top, \quad \rho(0) = [\rho_\circ, 0]^\top \tag{4}$$

where

$$\mathbb{J} = N_r \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

satisfy $\rho(t) \in \mathbb{S}^{\rho_\circ}$ for all $t \geq 0$ and are given by $\rho(t) = \varrho(\theta(t))$ where $\theta(t)$ is solution to (1c) *i.e.*,

$$\begin{aligned}\rho_1(t) &= \rho_\circ \cos\big(N_r[\theta(t)-\theta(0)]\big)\\ \rho_2(t) &= \rho_\circ \sin\big(N_r[\theta(t)-\theta(0)]\big).\end{aligned}$$

Furthermore, without loss of generality we consider that $\theta(0) = 0$. Then, the rotor dynamics (1b) takes the form

$$J\dot{\omega} = T_e(\rho, x) - T_L \tag{5}$$

in which T_L is constant and it is assumed that the mechanical torque is now expressed as a function of ρ that is,

$$T_e(\rho, x) = \frac{1}{2} x^\top K(\rho) x \tag{6}$$

hence, although T_e and $\mathbf{T}_e$ are different functions $T_e(\rho(\theta), x)$ and $\mathbf{T}_e(\theta, x)$ represent the same quantity. The matrix $K(\rho) = \ell_1 K'(\rho)$ with

$$K'(\rho) = N_r \begin{bmatrix} \rho_2 & 0 & 0 \\ 0 & \frac{1}{2}\left(\rho_2 - \sqrt{3}\rho_1\right) & 0 \\ 0 & 0 & \frac{1}{2}\left(\rho_2 + \sqrt{3}\rho_1\right) \end{bmatrix}. \tag{7}$$

In the new coordinates the stator equation becomes

$$L(\rho)\dot{x} + K(\rho)\omega x + Rx = u \tag{8}$$

where $L(\rho) = \ell_0 I + \ell_1 L'(\rho)$ and

$$L'(\rho) = \begin{bmatrix} -\rho_1 & 0 & 0 \\ 0 & -\frac{1}{2}\left(\rho_1 + \sqrt{3}\rho_2\right) & 0 \\ 0 & 0 & -\frac{1}{2}\left(\rho_1 - \sqrt{3}\rho_2\right) \end{bmatrix}. \tag{9}$$

It is clear that there exist positive constants ℓ_m, ℓ_M, k_m and k_M such that

$$\ell_m \leq |L(\rho)| \tag{10a}$$

$$\ell_M |\rho_1 - \rho_2| \geq \max\{|L(\rho_1 - \rho_2)|, |L(\rho_1) - L(\rho_2)|\} \tag{10b}$$

$$k_M |\rho_1 - \rho_2| \geq \max\{|K(\rho_1 - \rho_2)|, |K(\rho_1) - K(\rho_2)|\}. \tag{10c}$$

Thus, under the conditions described in Section II-A, the motor dynamics is defined by Equations (4), (5) and (8). The advantage of the rotor dynamics model (4), (5) is that it is linear in the new 'position' variables, ρ.

For the purpose of tracking control we introduce a reference oscillator dynamics for (4). Given a desired constant reference ω^*, we introduce θ^* as the angular position reference for θ that is, $\dot{\theta}^* = \omega^*$ and the reference oscillator dynamics

$$\dot{\rho}^* = \omega^* \mathbb{J} \rho^*, \quad \rho^*(0) = [\rho_\circ^*, 0]^\top \tag{11}$$

where the initial condition $\rho_\circ^* \in \mathbb{R}_+$ is a free design parameter. The solutions to (11) which define the angular reference trajectories, are

$$\rho^*(t) = \rho_\circ^* \begin{bmatrix} \cos\left(N_r[\theta^*(t) - \theta^*(0)]\right) \\ \sin\left(N_r[\theta^*(t) - \theta^*(0)]\right) \end{bmatrix} \tag{12}$$

where $\theta^*(t) = \omega^* t + \theta^*(0)$ and the initial reference angular position $\theta^*(0) \in [-\pi, \pi]$; without loss of generality we fix $\theta^*(0) = 0$. Note that $\rho^*(t) \in \mathbb{S}^{\rho_\circ^*}$ for all $t \geq 0$.

B. *Torque sharing*

This technique is used to induce a virtual control input into the mechanical equation (5). Ideally, the virtual control input enters through the mechanical torque. That is, given a control law T_d, one must solve the equation

$$\frac{\ell_1 T_e^*(\rho, x^*)}{J} = T_d \tag{13}$$

for the current reference x^*. Then, Equation (5) may be equivalently written as

$$J\dot{\omega} = J T_d - T_L + [T_e - \ell_1 T_e^*] \tag{14}$$

which for control purposes, may be viewed as a nominal system $\dot{\omega} = T_d - T_L/J$ perturbed by the term $[T_e - \ell_1 T_e^*]$. By design, T_d is such that $\omega \to \omega^*$ provided that $[T_e - \ell_1 T_e^*] \equiv 0$ and $[T_e - \ell_1 T_e^*]$ vanishes provided that current reference trajectories x^* are asymptotically tracked.

Clearly, the difficulty to solve (13) relies on the fact that T_e is quadratic in x^*. The torque sharing approach as used in [12], [5], [13], exploits the fact that the mechanical torque T_e corresponds to the sum of torques due to each phase therefore, we define

$$T_e^* = \frac{1}{2}\Big[K_1'(\rho^*)x_1^{*2} + K_2'(\rho^*)x_2^{*2} + K_3'(\rho^*)x_3^{*2}\Big].$$

and we solve (13) for x_j^* to obtain

$$x_j^* = \begin{cases} \left[\dfrac{2J}{\ell_1}\right]^{1/2}\left[\dfrac{m_j(\rho^*)T_d}{K_j'(\rho^*)}\right]^{1/2} & \text{if } K_j'(\rho^*) \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

where the functions m_j ensure that x_j exists for any ρ^* and T_d. That is, depending on the current phase of the reference model, the function m_j ensures that the respective signs of the numerator and of the denominator in the previous expression are equal for *at least one* $j \in \{1, 2, 3\}$. To that end, we define the sets

$$\begin{aligned} \Theta_j^+ &= \big\{\rho^* \in \mathbb{S}^{\rho_\circ^*} : K_j'(\rho^*) \geq 0\big\} \\ \Theta_j^- &= \big\{\rho^* \in \mathbb{S}^{\rho_\circ^*} : K_j'(\rho^*) < 0\big\} \end{aligned}$$

where the superscripts $^+$ and $^-$ stand for required positive and negative torque respectively. Accordingly, given T_d, we define

$$m_j(\rho^*) = \begin{cases} m_j^+(\rho^*) & \text{if} \quad T_d \geq 0, \\ m_j^-(\rho^*) & \text{if} \quad T_d < 0. \end{cases}$$

where

$$\begin{array}{llll} m_j^+(\rho^*) > 0 & \forall \rho^* \in \Theta^+, & m_j^+(\rho^*) = 0 & \forall \rho^* \in \Theta^-, \\ m_j^-(\rho^*) > 0 & \forall \rho^* \in \Theta^-, & m_j^-(\rho^*) = 0 & \forall \rho^* \in \Theta^+. \end{array}$$

Moreover, we impose that

$$\sum_{j=1}^{3} m_j^+(\rho^*) = 1, \qquad \sum_{j=1}^{3} m_j^-(\rho^*) = 1.$$

so we have

$$T_d = m_1(\rho^*)T_d + m_2(\rho^*)T_d + m_3(\rho^*)T_d \tag{16}$$

and (13) holds.

Remark 3.1: In the previously cited references the torque-sharing technique is implemented using θ instead of ρ^* that is, the reference x_j^* depends on the current phase of the motor and not of the *reference* model.

For clarity of exposition, we divide the rest of the paper in three parts that are coherent with the control approach described above. First, we discuss the control of the rotor dynamics (design of T_d) then, we present a tracking ($x \to x^*$) control law u for the stator dynamics. Finally, using a small-gain argument we establish that the the interconnection of the two subsystems schematically represented in Figure 1, is exponentially stable.

IV. Rotor robust state-feedback control

We present two preliminary results on robust state-feedback control of the rotor dynamics. In the first case, we establish a result of practical stability with respect to the uncompensated constant disturbance induced by the load-torque; in the second case, we add an integrator establish global exponential stability. In both scenarios we recover a property of input-to-state stability with respect to external inputs. This is significant to analyze the stability of the system interconnected with the stator dynamics.

A. Without load compensation

Let $\nu^* = \frac{T_L}{J}$ then, the rotor model is given by

$$\dot{\omega} = \frac{T_e}{J} - \nu^* \tag{17a}$$
$$\dot{\rho} = \omega \mathbb{J} \rho. \tag{17b}$$

Define $e_\rho := \rho - \rho^*$ and $e_\omega := \omega - \omega^*$ then, according to the policy described in Section III-B, we pose the state-feedback control law

$$T_d = -k_d e_\omega - k_p \rho^{*\top} \mathbb{J}^\top e_\rho + \nu + \dot{\omega}^*. \tag{18}$$

Define further, $\tilde{\nu} := \nu - \nu^*$ and add $T_d - \ell_1 T_e^*/J$ to the right-hand side of Equation (17a). Then, the latter may be rewritten as

$$\dot{e}_\omega = -k_d e_\omega - k_p \rho^{*\top} \mathbb{J}^\top e_\rho + \tilde{\nu} + \Delta_1(t, e_x, e_\rho) \tag{19a}$$
$$\Delta_1(t, e_x, e_\rho) = \frac{\ell_1}{2J} \left[x^\top K'(e_\rho) x - x^{*\top} K'(\rho^*) x^* + x^\top K'(\rho^*) x \right]. \tag{19b}$$

Subtracting (11) from (17b) and defining $v = \Delta_1 + \tilde{\nu}$, the mechanical error dynamics becomes

$$\dot{e}_\omega = -k_d e_\omega - k_p \rho^*(t)^\top \mathbb{J}^\top e_\rho + v \tag{20a}$$
$$\dot{e}_\rho = e_\omega \mathbb{J} \rho^*(t) + \omega \mathbb{J} e_\rho \tag{20b}$$

which may be viewed as a non-autonomous periodic system perturbed by the input v. The interest of this observation relies on the following statement.

Proposition 4.1 (GES by state-feedback, no load): Let v be bounded. Then, the system (20) is input-to-state-stable with respect to the input v and the map $v \to e_\omega$ is output-strictly passive. In addition, in the case that $v \equiv 0$ the origin $(e_\rho, e_\omega) = (0, 0)$ of (20) is globally exponentially stable.

Proof. Consider the positive definite radially unbounded function V_{c1},

$$V_{c1}(e_\omega, e_\rho) = \frac{1}{2} \left(e_\omega^2 + k_p |e_\rho|^2 \right) \tag{21}$$

whose time derivative along the trajectories of (20) yields

$$\dot{V}_{c1}(e_\omega, e_\rho) = -k_d e_\omega^2 + e_\omega v \,. \tag{22}$$

Output strict passivity of the map $v \mapsto e_\omega$ follows by integrating on both sides of (22) since $V_{c1} \geq 0$. The proof of global asymptotic stability under the condition $v \equiv 0$, follows invoking Lasalle's theorem for periodic systems –see *e.g.* [17, Theorem 5.3.79]: note that $e_w = 0$ implies that $\dot{V}_{c1} = 0$ and the only solution of $k_p \rho^*(t)^\top \mathbb{J}^\top e_\rho = 0$ for any t, is $e_\rho = 0$ that is, the origin is the largest invariant set contained in $\{\dot{V}_{c1} = 0\}$. Global exponential stability is established invoking standard results from adaptive control literature, observing that $\mathbb{J}\rho^*(t)$ is persistently exciting that is, there exist T_c and $\mu_c > 0$ such that

$$\int_t^{t+T_c} \mathbb{J} \rho^*(\tau) \rho^*(\tau)^\top \mathbb{J}^\top d\tau \geq \mu_c I. \tag{23}$$

As a matter of fact, (23) holds with $T_c = \pi / N_r \omega^*$ and $\mu_c = |\rho_\circ^*|^2 N_r^2 T_c / 2$ –see Appendix IX-A. Input-to-state stability follows from the following statement.

Lemma 4.1: Let $\varepsilon_1 \in (0, 1)$ be a small parameter to be defined, let $i \in \{1, \ldots 6\}$, $\lambda_i \geq 0$ be such that $\sum_{i=1}^6 \lambda_i = 1$, $k_{di} := \lambda_i k_d$ and $\bar{k}_{dj} := \sum_{i=j}^6 k_{di}$; similarly for k_p. Define the functions V_{c2}, V_{c3} by

$$V_{c2}(t, e_\omega, e_\rho) = \varepsilon_1 e_\omega \rho^*(t)^\top \mathbb{J}^\top e_\rho \tag{24a}$$
$$V_{c3}(t, e_\rho) = -e_\rho^\top \int_t^{t+T_c} e^{(t-\tau)} \mathbb{J} \rho^*(\tau) \rho^*(\tau)^\top \mathbb{J}^\top d\tau e_\rho \,. \tag{24b}$$

Then, if

$$|\rho_\circ^*| \geq \frac{1}{2} e^{\pi / N_r \omega^*} \tag{25}$$

we have

$$\sum_{i=1}^{3}\dot{V}_{ci} \;\leq\; -\bar{k}_{d3}e_\omega^2 - \frac{N_r\pi}{2\omega^*}\left|\rho_\circ^*\right|\left(\left|\rho_\circ^*\right|e^{-\pi/\omega^*} - \frac{N_r}{2}\right)\left|e_\rho\right|^2$$
$$-\varepsilon_1\bar{k}_{p3}\left[\rho^{*\top}\mathbb{J}^\top e_\rho\right]^2 + \left(\varepsilon_1\left[\rho^{*\top}\mathbb{J}^\top e_\rho\right] + e_\omega\right)v. \tag{26}$$

The proof of Lemma 4.1 is included in Appendix IX-B. Input-to-state stability with respect to the input v follows remarking that $\sum_{i=1}^{3}V_{ci}$ is an ISS-Lyapunov function; indeed, it is enough to choose a constant α sufficiently small such that $|v| \leq \alpha\left|e_\omega, e_\rho\right|$ implies that $\sum_{i=1}^{3}\dot{V}_c$ is negative definite. ■

B. With compensation of unknown load

Proposition 4.1 establishes global exponential stability for the system without load torque. As a byproduct, the system is robust with respect to additive disturbances such as torque-load uncertainty ($\tilde{\nu}$ = const.). By exploiting the passivity of (20) we add a second loop which we close with integral action, to compensate for $\tilde{\nu}$. That is, let the variable ν in (18) be defined by

$$\dot{\nu} = -k_i\left(e_\omega + \varepsilon_1\left[\rho^{*\top}\mathbb{J}^\top e_\rho\right]\right), \quad k_i > 0, \tag{27}$$

then, the map $\left(e_\omega + \varepsilon_1\left[\rho^{*\top}\mathbb{J}^\top e_\rho\right]\right) \mapsto \nu$ is passive, the passivity and robustness properties of (20) are conserved.

Proposition 4.2 (GES by state-feedback, with load compensation): The system (20) with $v = \tilde{\nu} + \Delta_1$ is input-to-state-stable with respect to Δ_1 and the map $\Delta_1 \mapsto \left(e_\omega + \varepsilon_1\left[\rho^{*\top}\mathbb{J}^\top e_\rho\right]\right)$ is output-strictly passive. Moreover, if $\Delta_1 \equiv 0$ that is if $v = \tilde{\nu}$, then the origin $(e_\rho, e_\omega, \tilde{\nu}) = (0, 0, 0)$ of (20) is globally exponentially stable for appropriate values of the gains k_p, k_d and k_i.

Proof. Consider the system (20) with $v = \tilde{\nu} + \Delta_1$ and the function

$$V_{c4}(\tilde{\nu}) := \frac{1}{2k_i}\tilde{\nu}^2. \tag{28}$$

The total time derivative of $\sum_{i=1}^{4}V_{ci}$ along the trajectories of (20), and

$$\dot{\tilde{\nu}} = -k_i\left(e_\omega + \varepsilon_1\left[\rho^{*\top}\mathbb{J}^\top e_\rho\right]\right), \quad k_i > 0 \tag{29}$$

satisfies (26) with $v = \Delta_1$. Integrating the resulting expression of $\sum_{i=1}^{4}\dot{V}_{ci}$ on both sides, we see that the map $\Delta_1 \mapsto \left(e_\omega + \varepsilon_1\left[\rho^{*\top}\mathbb{J}^\top e_\rho\right]\right)$ is output-strictly passive. Furthermore, if $\Delta_1 \equiv 0$ global asymptotic stability follows invoking Lasalle's principle, as in the proof of Proposition 4.1.

Now we proceed to show that $\sum_{i=1}^{5}V_{ci}$ with

$$V_{c5}(e_\rho, e_\omega, \tilde{\nu}) := -\varepsilon_3\tilde{\nu}e_\omega - \frac{1}{2}\varepsilon_1\varepsilon_3k_i\left|e_\rho\right|^2 \tag{30}$$

qualifies as an ISS-Lyapunov function. The total time derivative of V_{c5} along the trajectories generated by (20), (29) yields

$$\begin{aligned}\dot{V}_{c5} \;=\; & \varepsilon_3k_i\left(e_\omega + \varepsilon_1\left[\rho^{*\top}\mathbb{J}^\top e_\rho\right]\right)e_\omega \\ & -\varepsilon_3\tilde{\nu}^2 - \varepsilon_3\tilde{\nu}\left(-k_de_\omega - k_p\rho^{*\top}\mathbb{J}^\top e_\rho + \Delta_1\right) \\ & -\varepsilon_1\varepsilon_3k_ie_\rho^\top\left[\omega\mathbb{J}e_\rho + e_\omega\mathbb{J}\rho^*\right].\end{aligned} \tag{31}$$

Adding $\dot{V}_{c4}$ and the latter to (26), we obtain

$$\sum_{i=1}^{5}\dot{V}_{ci} \;\leq\; -[\bar{k}_{d4} - \varepsilon_3k_i]e_\omega^2 - \frac{N_r\pi}{2\omega^*}\left|\rho_\circ^*\right|\left(\left|\rho_\circ^*\right|e^{-\pi/N_r\omega^*} - \frac{N_r}{2}\right)\left|e_\rho\right|^2 - \varepsilon_1\bar{k}_{p4}\left[\rho^{*\top}\mathbb{J}^\top e_\rho\right]^2 - \frac{\varepsilon_3}{2}\tilde{\nu}^2$$
$$-\,\delta_2 + \Delta_1\left(\varepsilon_1\left[\rho^{*\top}\mathbb{J}^\top e_\rho\right] + e_\omega - \varepsilon_3\tilde{\nu}\right) \tag{32}$$

where we recall (see Lemma 4.1) that $\bar{k}_{dj} = \sum_{i=j}^{6} k_{di}$, $k_{di} = \lambda_i k_d$ (similarly for k_p) and we defined

$$\delta_2 := \frac{1}{2} \begin{bmatrix} e_\omega \\ \rho^{*\top} \mathbb{J}^\top e_\rho \\ \tilde{\nu} \end{bmatrix} \begin{bmatrix} 2k_{d3} & 0 & \varepsilon_3 k_d \\ 0 & 2\varepsilon_1 k_{p3} & \varepsilon_3 k_p \\ \varepsilon_3 k_d & \varepsilon_3 k_p & \varepsilon_3 \end{bmatrix} \begin{bmatrix} e_\omega \\ \rho^{*\top} \mathbb{J}^\top e_\rho \\ \tilde{\nu} \end{bmatrix}.$$

Let ε_3 satisfy

$$\min \left\{ \frac{k_{d4}}{k_i}, \frac{\lambda_3}{k_d}, \frac{\varepsilon_1 \lambda_3}{k_p} \right\} \geq \varepsilon_3 \tag{33}$$

then, $\delta_2 \geq 0$ and $\sum_{i=1}^{5} V_{ci}$ is an ISS-Lyapunov function for system (20) with $v = \tilde{\nu} + \Delta_1$ and (29), with respect to the input Δ_1. Furthermore, if $\Delta_1 \equiv 0$, $\sum_{i=1}^{5} \dot{V}_{ci}$ is bounded by a quadratic negative definite function of the state; global exponential stability follows invoking standard Lyapunov theory. ■

V. Stator robust state-feedback control

In the previous section we established input-to-state stability for the rotor dynamics with respect to inputs Δ_1 which vanish with $e_x = x - x^*$. In this section we focus on the tracking control of the stator dynamics that is, the control goal is to make $x \to x^*$ where $x^* := [x_1^* \, x_2^* \, x_3^*]^\top$ and the latter is defined by (15). The controller that we propose establishes global exponential stability in the case of perfect velocity tracking ($e_\omega = 0$) and input to state stability with respect to external inputs which vanish as $e_w \to 0$.

For Equation (8) we introduce the control law

$$u^*(t,x) := L(\rho^*)\dot{x}^* + K(\rho^*)\omega^* x + Rx^* - k_{px} e_x \tag{34}$$

where k_{px} is shorthand notation for $k_{px}(t, |e_x|)$ and is defined by a continuous function $k_{px} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that $k_{px}(t, \cdot)$ is non-decreasing. Note that $\dot{x}^*$ is a function of time, $\dot{T}_d$ and T_d which depend only on measured states and computed quantities. Indeed, defining

$$\sigma_j(\rho^*) := \frac{m_j(\rho^*)}{K_j'(\rho^*)}$$

we have, after (15),

$$\dot{x}_j^* = \begin{cases} \left[\dfrac{2J}{\ell_1} \right]^{1/2} [\sigma_j(\rho^*) T_d]^{-1/2} \left[\dot{\sigma}_j T_d + \sigma_j \dot{T}_d \right] & \text{if } K_j'(\rho^*) \neq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{35}$$

Applying $u = u^*$ into (8) we see that

$$L(\rho)\dot{e}_x + [R + k_{px}] e_x = \Delta_2(t, e_\rho, e_x, \dot{x}^*) \tag{36a}$$

$$\Delta_2(t, e_\rho, e_x, \dot{x}^*) = -\left[L'(e_\rho)\dot{x}^* + K(e_\rho)\omega^* x + K(\rho) e_\omega x \right] \tag{36b}$$

and, from (10) we have

$$|\Delta_2| \leq \left[\ell_M |\dot{x}^*| + k_M \omega^* |x| \right] |e_\rho| + k_M |\rho_\circ^*| \, |x| \, |e_\omega| . \tag{37}$$

That is, the origin $\{e_x = 0\}$ of the the stator closed-loop system is exponentially stable in the case that the rotor controller achieves perfect velocity tracking. Global exponential stability for (36), implies local input to state stability; the global property is established next.

Proposition 5.1: Let $\rho_\circ = \rho_\circ^*$ and let[1]

$$u = u^* - \left[\ell_M |\dot{x}^*| + k_M \omega^* |x| \right] |e_\rho| \operatorname{sgn}(e_x) . \tag{38}$$

Assume further that

$$k_{px} := k_{px1} + \frac{1}{2} \left[k_M |\rho_\circ^*| \, |x| \right]^2, \quad k_{px1} > 0 \tag{39}$$

[1]As usual, the sign function is defined as $\text{sign}(0) \in [-1, 1]$ and $\text{sign}(x) = \text{abs}(x)/x$ if $x \neq 0$. By an abuse of notation, the vector $\text{sgn}(e_x) = \text{col}[\text{sign}(e_{xi})]$.

then, the closed-loop system (8) with (38) is input-to-state stable from the input e_ω. Moreover, in the case that $|\Delta_2| \equiv 0$, the origin $\{e_x = 0\}$ is globally exponentially stable with $u = u^*$ and $k_{px} := k_{px1}$.

Proof. The total time derivative of

$$V_{c6}(e_x) := \frac{1}{2}|e_x|^2 \tag{40}$$

along the closed-loop trajectories[2] yields

$$\dot{V}_{c6} \leq -[R + k_{px}]|e_x|^2 + \Big[k_M |\rho_\circ^*|\,|x|\Big]|e_\omega|\,|e_x| \tag{41}$$

which, in view of (39), implies that

$$\dot{V}_{c6} \leq -[R + k_{px1}]|e_x|^2 + \frac{1}{2}|e_\omega|^2 \tag{42}$$

hence, V_{c6} is an ISS-Lyapunov function for the stator closed-loop system. The proof of the second statement follows directly observing that $|\Delta_2| \equiv 0$ implies that $\dot{V}_{c6} \leq -[R + k_{px}]|e_x|^2$. ∎

VI. Robust control of switched-reluctance motor

Now we present our main results, we establish global exponential stability for the controlled switched-reluctance via state feedback. We also establish that the interconnection of the two control loops for the rotor dynamics and the stator dynamics, remains input-to state stable with respect to external inputs.

Proposition 6.1: Consider the system (1) under the assumptions described in Section II-A in closed loop with the controller defined by (38), (35) with $\rho_\circ = \rho_\circ^*$ and (18), (15). Let k_{px} be given by (39) where

$$k_{px1} \geq \frac{1}{2}\Big[k_M |\rho_\circ^*| \left(|x^*| + |x|\right)\Big]^2 (\varepsilon_1 + 2\varepsilon_3 + 1) \tag{43}$$

where ε_1 and ε_3 are small positive constants and let (25) hold. Then, the origin of the closed-loop system is globally exponentially stable.

Proof. The motor model under the conditions described in Section II-A corresponds to the equations (8) and (17). Therefore, the closed-loop system corresponds to (19), (20b), (29) and (36). The term Δ_1 in (19b) satisfies $2J\Delta_1 = x^\top K(e_\rho)x + e_x^\top K(\rho^*)x^* + e_x^\top K(\rho^*)x$ hence, using $|K(\rho^*)| \leq k_M |\rho_\circ^*|$ we see that

$$|\Delta_1| \leq \frac{k_M}{J}|\rho_\circ^*|\,|e_x|\Big[|x^*| + |x|\Big]. \tag{44}$$

In view of the latter, (32), (33) and (42), it follows that the total time derivative of $V_c := \sum_{i=1}^{6} V_{ci}$ satisfies

$$\begin{aligned}
\dot{V}_c \;\leq\;& -[\bar{k}_{d5} - 0.5]e_\omega^2 - \left[\frac{N_r\pi}{2\omega^*}|\rho_\circ^*|\left(|\rho_\circ^*|\,e^{-\pi/N_r\omega^*} - \frac{N_r}{2}\right)\right]|e_\rho|^2 - \varepsilon_1\bar{k}_{p4}\Big[\rho^{*\top}\mathbb{J}^\top e_\rho\Big]^2 - \frac{\varepsilon_3}{2}\tilde{\nu}^2 \\
& -[R + k_{px1}]|e_x|^2 + \frac{k_M}{J}|\rho_\circ^*|\,|e_x|\Big[|x^*| + |x|\Big]\left(\varepsilon_1\left|\rho^{*\top}\mathbb{J}^\top e_\rho\right| + |e_\omega| + \varepsilon_3|\tilde{\nu}|\right)
\end{aligned} \tag{45}$$

which, in virtue of the triangle inequality, (43) and provided that k_{p4}, $k_{d5} \geq 1$, implies that

$$\dot{V}_c \;\leq\; -W(e_x, e_\rho, e_\omega, \rho^{*\top}\mathbb{J}^\top e_\rho) \tag{46a}$$

$$\begin{aligned}
W \;\leq\;& \bar{k}_{d6}e_\omega^2 + \left[R + \frac{1}{2}k_{px1}\right]|e_x|^2 + \varepsilon_1\bar{k}_{p5}\Big[\rho^{*\top}\mathbb{J}^\top e_\rho\Big]^2 + \frac{\varepsilon_3}{4}\tilde{\nu}^2 \\
& + \left[\frac{N_r\pi}{2\omega^*}|\rho_\circ^*|\left(|\rho_\circ^*|\,e^{-\pi/N_r\omega^*} - \frac{N_r}{2}\right)\right]|e_\rho|^2
\end{aligned} \tag{46b}$$

hence, $\dot{V}_c$ is negative definite. Global exponential stability follows invoking standard Lyapunov theory.

Now, let v_m and v_s be bounded external inputs and reconsider (20a) with $v = \tilde{\nu} + \Delta_1 + v_m$ and let $u = u^* + v_s$. Then, from the previous development we obtain

$$\dot{V}_c \;\leq\; W(e_x, e_\rho, e_\omega, \rho^{*\top}\mathbb{J}^\top e_\rho) + e_x v_s + v_m\left(\varepsilon_1\Big[\rho^{*\top}\mathbb{J}^\top e_\rho\Big] + e_\omega - \varepsilon_3\tilde{\nu}\right) \tag{47}$$

That is, V_c qualifies as an ISS Lyapunov function for the closed-loop system with inputs v_s, v_m. ∎

[2]It is considered that solutions are defined in Filippov's sense.

VII. SIMULATION RESULTS

We have tested our main result in simulations using SIMULINK™of MATLAB™. The parameters of the motors are $R = 5$, $l_0 = 0.030H$, $J = 0.01kg - m^2$ and $N_r = 8$. The applied load-torque is constant and equals 0.1[Nm]. The control gains are fixed to $k_p = 2000$, $k_i = 5e - 4$, $k_d = 15000$ and $k_{px} = 50$.

The reference consists in a smooth function which gradually increases from an initial value (here, 0[rad/s]) to the constant desired speed and is defined using the function, which in turn is set up in 50[rad/s].

$$\omega^*(t) = \Big(\frac{1 - e^{-\alpha(t-T)}}{1 + e^{-\alpha(t-T)}} + 1\Big)\Big(\frac{\omega_f^* - \omega_0^*}{2}\Big) + \omega_0^*. \tag{48}$$

The initial velocity ω_0 is set to 25[rad/s] that is, 50% of initial error with respect to the set-point reference.

The construction of the functions m_j is as follows. First, we define θ^* as a function of ρ^* *i.e.*

$$\theta^*(\rho^*) = \begin{cases} \beta(\rho^*) & \text{if } \rho_2^* > \rho_1^* > 0 \\ \beta(\rho^*) + \frac{\pi}{4} & \text{if } \rho_1^* > \rho_2^* > 0 \text{ or } \rho_1^* > 0 > \rho_2^* \\ \beta(\rho^*) + \frac{\pi}{8} & \text{if } \rho_1^* < 0 \end{cases} \tag{49}$$

where $\beta(\rho^*) = \dfrac{tg^{-1}\,(\rho_2^*/\rho_1^*) - 2\pi/N_r}{N_r}$. Then, $\theta^* \in [0, 2\pi/N_r]$ is used in the construction of the auxiliary functions $\tilde{m}_j^+(\cdot)$ and $\tilde{m}_j^-(\cdot)$,

$$\tilde{m}_j^+(\theta^*) = \begin{cases} f(\theta_1^*) & \text{if } 0 < \theta_1^* \leq \frac{\pi}{3N_r} \\ 1 & \text{if } \frac{\pi}{3N_r} < \theta_1^* \leq \frac{2\pi}{3N_r} \\ 1 - f(\theta_1^* - 2\pi/N_r) & \text{if } \frac{2\pi}{3N_r} < \theta_1^* \leq \frac{\pi}{N_r} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\tilde{m}_j^-(\theta^*) = \begin{cases} f(\theta_j^* - \frac{\pi}{N_r}) & \text{if } \frac{\pi}{N_r} < \theta_1^* \leq \frac{4\pi}{3N_r} \\ 1 & \text{if } \frac{4\pi}{3N_r} < \theta_1^* \leq \frac{5\pi}{3N_r} \\ 1 - f(\theta_1^* - 5\pi/3N_r) & \text{if } \frac{5\pi}{3N_r} < \theta_1^* \\ 0 & \text{otherwise} \end{cases}$$

with $\theta_1^* = \theta^*$, $\theta_2^* = \theta^* - 2\pi/3N_r$, $\theta_3^* = \theta^* + 2\pi/3N_r$. Finally, $m_j(\rho^*)$ is obtained from

$$m_j(\rho^*) = \begin{cases} \tilde{m}_j^+(\theta^*(\rho^*)) & \text{if } T_d \geq 0 \\ \tilde{m}_j^-(\theta^*(\rho^*)) & \text{if } T_d < 0 \end{cases} \tag{50}$$

in Fig. (2) is noticed that for each $j \in \{1, 2, 3\}$, $m_j^+(\rho^*) = \tilde{m}_j^+(\theta^*(\rho^*))$ is larger than zero only when $K_j(\rho^*) > 0$ and it equals zero when $K_j(\rho^*) < 0$. Similarly $m_j^-(\rho^*) = \tilde{m}_j^-(\theta^*(\rho^*))$ is larger than zero only when $K_j(\rho^*) < 0$, the last guarantees that it is always possible to compute x^* as is expressed in (15). See Figure 2.

The corresponding reference currents are depicted in the zoomed window showed in Figure 3 against the actual currents and the commutation functions m_j along trajectories. The good current tracking performance as well as the commutation among the three phases of the reference oscillator are clearly appreciated.

The voltage inputs for the three phases are showed in Figure 4.

Finally, we show the good velocity tracking performance in Figure 5 both for $\rho(t)$ and $\omega(t)$. Note that in both cases the errors $e_\rho(t)$ converge to zero asymptotically that is, the rotor synchronizes with the virtual rotor, generated by the reference oscillator.

Simulation results are encouraging to pursue this avenue towards the solution of full-sensorless control via certainty-equivalence control.

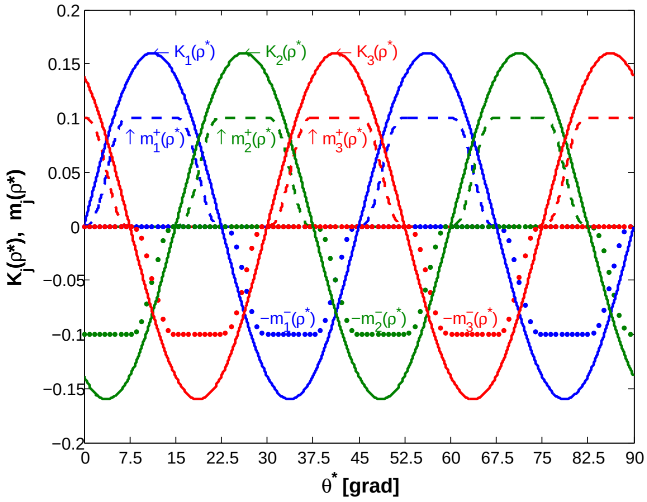

Fig. 2. Construction of functions $m_j(\rho^*)$

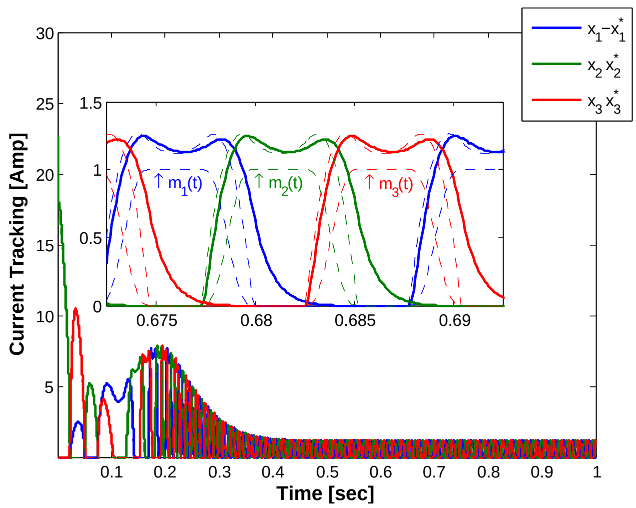

Fig. 3. Three phases currents and their references

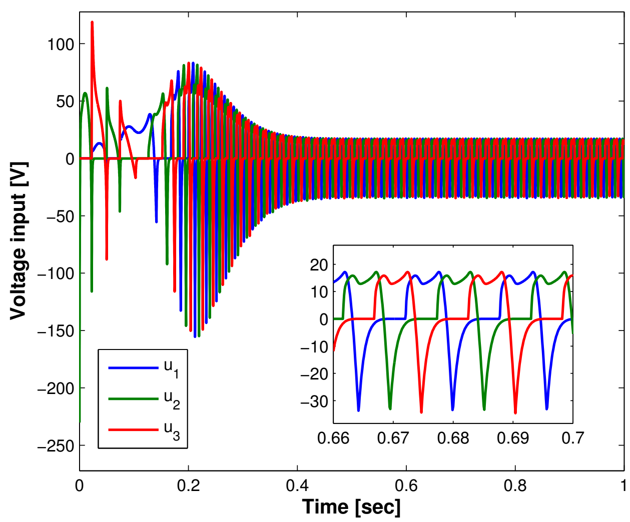

Fig. 4. Voltage control inputs

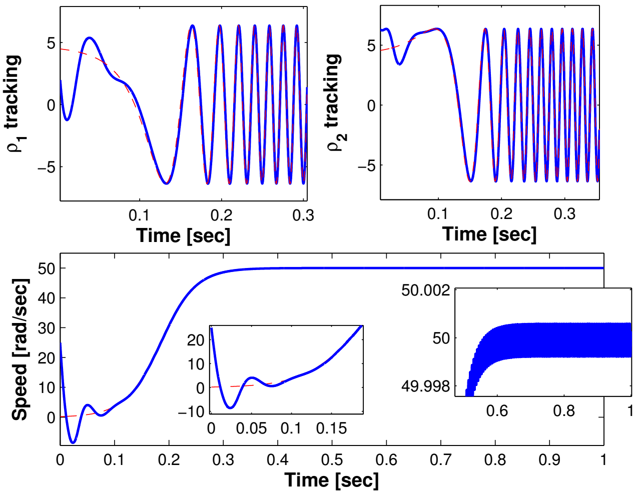

Fig. 5. Velocity $\omega(t)$ and angular position in polar coordinates $\rho(t)$

VIII. Conclusions

We have presented a control approach to the robust stabilization of the switched-reluctance motor. The control approach that we presented consists in stabilizing separately the stator and the rotor dynamics. We have established global exponential stability. In addition, our control scheme has the special feature of being tailored to be implemented as a certainty-equivalence controller, with a state estimator. The design of the latter is under current research.

Acknowledgements

The work of G. Espinosa is supported by DGAPA-UNAM under grant IN11121. This work was mostly carried out while A. Loria was visiting DGPA-UNAM supported by the latter institution. A. Loria's and E. Chumacero's research leading to these results has also received funding from the European Union Seventh Framework Program [FP7/2007-2013] under grant agreement n° 257462 HYCON2 Network of excellence. E. Chumacero benefits from a scholarship by CONACyT, Mexico.

References

[1] H. Vasquez, J. Parker, and T. Haskew, "Control of a 6/4 switched reluctance motor in a variable speed pumping application," *Mechatronics*, vol. 15, no. 9, pp. 1061–1071, 2005.

[2] F. Daldaban and N. Ustkoyuncu, "A novel linear switched reluctance motor for railway transportation systems," *Energy Conversion and Management*, vol. 51, no. 3, pp. 465 – 469, 2010.

[3] D. Cajander and H. Le-Huy, "Design and optimization of a torque controller for a switched reluctance motor drive for electric vehicles by simulation," *Mathematics and Computers in Simulation*, vol. 71, no. 4–6, pp. 333–344, 2006.

[4] J. Faiz and K. Moayed-Zadeh, "Design of switched reluctance machine for starter/generator of hybrid electric vehicle," *Electric Power Systems Research*, vol. 75, no. 2–3, pp. 153 – 160, 2005.

[5] G. Espinosa-Perez, P. Maya-Ortiz, M. Velasco-Villa, and H. Sira-Ramirez, "Passivity-based control of switched reluctance motors with nonlinear magnetic circuits," *Control Systems Technology, IEEE Transactions on*, vol. 12, pp. 439 – 448, may 2004.

[6] H. Hannoun, M. Hilairet, and C. Marchand, "High performance current control of a switched reluctance machine based on a gain-scheduling pi controller," *Control Engineering Practice*, vol. 19, no. 11, pp. 1377 – 1386, 2011.

[7] H. Chen and J. Gu, "Implementation of the three-phase switched reluctance machine system for motors and generators," *Mechatronics, IEEE/ASME Transactions on*, vol. 15, pp. 421 –432, june 2010.

[8] C. Mademlis and I. Kioskeridis, "Gain-scheduling regulator for high-performance position control of switched reluctance motor drives," *Industrial Electronics, IEEE Transactions on*, vol. 57, pp. 2922 –2931, sept. 2010.

[9] H. Gao, F. Salmasi, and M. Ehsani, "Inductance model-based sensorless control of the switched reluctance motor drive at low speed," *Power Electronics, IEEE Transactions on*, vol. 19, pp. 1568 – 1573, nov. 2004.

[10] S. Hossain, I. Husain, H. Klode, B. Lequesne, A. Omekanda, and S. Gopalakrishnan, "Four-quadrant and zero-speed sensorless control of a switched reluctance motor," *Industry Applications, IEEE Transactions on*, vol. 39, pp. 1343 – 1349, sept.-oct. 2003.

[11] H. Peyrl, G. Papafotiou, and M. Morari, "Model predictive torque control of a switched reluctance motor," in *Industrial Technology, 2009. ICIT 2009. IEEE International Conference on*, pp. 1 –6, feb. 2009.

[12] D. Taylor, "Pulse-width modulated control of electromechanical systems," *IEEE Trans. Automat. Contr.*, vol. AC-37, pp. 524–528, 1992.

[13] M. Ilic-Spong, R. Marino, S. Peresada, and D. Taylor, "Feedback linearizing control of switched reluctance motors," *IEEE Trans. Automat. Contr.*, vol. AC-32, pp. 371–379, 1987.

[14] R. Krishnan, *Switched Reluctance Motor Drives*. CRC Press, 2001.

[15] A. Cheok and Y. Fukuda, "A new torque and flux control method for switched reluctance motor drives," *Power Electronics, IEEE Transactions on*, vol. 17, pp. 543 –557, jul 2002.

[16] R. Ortega, L. Praly, A. Astolfi, J. Lee, and K. Nam, "Estimation of rotor position and speed of permanent magnet synchronous motors with guaranteed stability," *Control Systems Technology, IEEE Transactions on*, vol. 19, pp. 601 –614, may 2011.

[17] M. Vidyasagar, *Nonlinear systems analysis*. New Jersey: Prentice Hall, 1993.

IX. Appendix

A. Properties of the reference oscillator

By design $\theta^*(0) = 0$ therefore $\theta^*(t) = \omega^* t$ and we have $\mathbb{J}\rho^*(\tau)\rho^*(\tau)^\top \mathbb{J}^\top = N_r^2 \left|\rho_\circ^*\right|^2 \Psi(\tau)$ where the matrix

$$\Psi(\tau) := \begin{bmatrix} \sin(N_r\theta^*(\tau))^2 & -\sin(N_r\theta^*(\tau))\cos(N_r\theta^*(\tau)) \\ -\sin(N_r\theta^*(\tau))\cos(N_r\theta^*(\tau)) & \cos(N_r\theta^*(\tau))^2 \end{bmatrix}$$

is periodic with period $\pi/N_r\omega^*$. Also, we have

$$\begin{aligned} \int_t^{t+\pi/N_r\omega^*} \sin(N_r\theta^*(\tau))^2 d\tau &= \frac{1}{N_r\omega^*}\int_{N_r\omega^* t}^{N_r\omega^*[t+\pi/N_r\omega^*]} \sin(\theta^*)^2 d\theta^* \\ &= \frac{1}{N_r\omega^*}\left[\frac{1}{2}\theta^* - \frac{1}{4}\sin(2\theta^*)\right]\Bigg|_{N_r\omega^* t}^{N_r\omega^*[t+\pi/N_r\omega^*]} \\ &= \frac{\pi}{2N_r\omega^*} - \frac{1}{4N_r\omega^*}\left[-\sin(2N_r\omega^*[t+\pi/N_r\omega^*])) + \sin(2N_r\omega^* t)\right] \\ &= \frac{\pi}{2N_r\omega^*} \end{aligned}$$

while a similar computation yields

$$\int_t^{t+\pi/N_r\omega^*} \cos(N_r\theta^*(\tau))^2 d\tau = \\ \frac{1}{N_r\omega^*}\left[\frac{1}{2}\theta^* + \frac{1}{4}\sin(2\theta^*)\right]\Bigg|_{N_r\omega^* t}^{N_r\omega^*[t+\pi/N_r\omega^*]} = \frac{\pi}{2N_r\omega^*}$$

On the other hand,

$$\int_t^{t+\pi/N_r\omega^*} \sin(N_r\theta^*(\tau))\cos(N_r\theta^*(\tau)) d\tau = \\ \int_t^{t+\pi/N_r\omega^*} \sin(2N_r\omega^*\tau) d\tau = 0$$

so finally, we obtain

$$\int_t^{t+\pi/N_r\omega^*} \Psi(\tau) d\tau = \frac{\pi}{2N_r\omega^*} I. \tag{51}$$

By the same reasoning we see that the product

$$\Psi\mathbb{J} = \begin{bmatrix} -\sin(N_r\theta^*)\cos(N_r\theta^*) & -\sin(N_r\theta^*)^2 \\ \cos(N_r\theta^*)^2 & \sin(N_r\theta^*)\cos(N_r\theta^*) \end{bmatrix} N_r$$

satisfies the following. The matrix

$$\Upsilon := \int_t^{t+\pi/N_r\omega^*} \Psi(\tau)\mathbb{J} d\tau = \begin{bmatrix} 0 & -\pi/2\omega^* \\ \pi/2\omega^* & 0 \end{bmatrix} \tag{52}$$

hence, it is skew-symmetric and

$$|\Upsilon| := \sqrt{\lambda_M\Big(\Upsilon^\top\Upsilon\Big)} = \pi/2\omega^*.$$

B. Proof of Lemma 4.1

The derivative of V_{c2} along the trajectories of (20) yields

$$\begin{aligned}\dot{V}_{c2} &= \varepsilon_1\Big[-k_d e_\omega - k_p\rho^{*\top}\mathbb{J}^\top e_\rho + v\Big]\rho^{*\top}\mathbb{J}^\top e_\rho \\ &\quad +\varepsilon_1 e_\omega\dot{\rho}^{*\top}\mathbb{J}^\top e_\rho + \varepsilon_1 e_\omega\rho^{*\top}\mathbb{J}^\top\Big[\mathbb{J}\rho^* e_\omega + \omega\mathbb{J}e_\rho\Big] \\ &= -\varepsilon_1 k_d e_\omega\rho^{*\top}\mathbb{J}^\top e_\rho - \varepsilon_1 k_p\Big[\rho^{*\top}\mathbb{J}^\top e_\rho\Big]^2 + \varepsilon_1 v\rho^{*\top}\mathbb{J}^\top e_\rho \\ &\quad +\varepsilon_1 e_\omega\rho^{*\top}\mathbb{J}^\top\mathbb{J}^\top e_\rho\omega^* + N_r^2\varepsilon_1|\rho_\circ^*|^2 e_w^2 + \varepsilon_1 e_w\rho^{*\top}\mathbb{J}^\top\mathbb{J}e_\rho\omega \\ &= -\varepsilon_1 k_p\Big[\rho^{*\top}\mathbb{J}^\top e_\rho\Big]^2 + N_r^2\varepsilon_1\Big[\rho^{*\top}e_\rho + |\rho_\circ^*|^2\Big]e_\omega^2 \\ &\quad -\varepsilon_1 k_d\Big[\rho^{*\top}\mathbb{J}^\top e_\rho\Big]e_\omega + \varepsilon_1\Big[\rho^{*\top}\mathbb{J}^\top e_\rho\Big]v\,. \end{aligned} \tag{53}$$

Next, we expose some properties of V_{c3}. Firstly, note that

$$V_{c3} = -e_\rho^\top\left[\int_t^{t+T_c} e^{(t-\tau)}\Psi(\tau)d\tau\right]e_\rho N_r^2|\rho_\circ^*|^2$$

and since $\Psi(\tau)\geq 0$ and $e^{(t-\tau)}\geq e^{(t-[t+T_c])}$ we have

$$V_{c3} \;\leq\; -e_\rho^\top\left[\int_t^{t+T_c}\Psi(\tau)d\tau\right]e_\rho N_r^2|\rho_\circ^*|^2 e^{-T_c}\,.$$

Then, setting $T_c = \pi/N_r\omega^*$ and using (51) we obtain

$$V_{c3} \leq -\left(\frac{N_r\pi}{2\omega^*}|\rho_\circ^*|^2 e^{-\pi/N_r\omega^*}\right)|e_\rho|^2 \tag{54}$$

in which $\rho_\circ^*$ is a design parameter. Furthermore, the total derivative of V_{c3} along the trajectories of (20) satisfies

$$\begin{aligned}\dot{V}_{c3} &\leq -\int_t^{t+T_c} e^{(t-\tau)}2e_\rho^\top\mathbb{J}\rho^*(\tau)\rho^*(\tau)^\top\mathbb{J}^\top\dot{e}_\rho d\tau \\ &\quad + \Big|\rho^*(t)^\top\mathbb{J}^\top e_\rho\Big|^2 + V_{c3}\,. \end{aligned} \tag{55}$$

Substituting (20b) in the first term we obtain

$$\begin{aligned} -\int_t^{t+T_c} & e^{(t-\tau)}2e_\rho^\top\mathbb{J}\rho^*(\tau)\rho^*(\tau)^\top\mathbb{J}^\top\dot{e}_\rho d\tau = \\ & -N_r^2|\rho_\circ^*|^2\int_t^{t+T_c} e^{(t-\tau)}e_\rho^\top\Psi(\tau)\mathbb{J}\rho^*(\tau)d\tau e_\omega \\ & -N_r^2|\rho_\circ^*|^2\omega e_\rho^\top\left(\int_t^{t+T_c} e^{(t-\tau)}\Psi(\tau)\mathbb{J}d\tau\right)e_\rho\,. \end{aligned} \tag{56}$$

Set $T_c = \pi/N_r\omega^*$. Then, we use (52) to see that the first term on the right-hand side of (56) is bounded by $\pi N_r^2|\rho_\circ^*|^3|e_\rho||e_\omega|/2\omega^*$ while the second term is bounded by

$$N_r^2|\rho_\circ^*|^2|\omega|\left|e_\rho^\top\int_t^{t+\pi/N_r\omega^*}\Psi(\tau)\mathbb{J}d\tau e_\rho\right| = 0;$$

see (52). We conclude that

$$\begin{aligned} \dot{V}_{c3} &\leq \left|\rho^*(t)^\top \mathbb{J}^\top e_\rho\right|^2 - \left(\frac{N_r\pi}{2\omega^*}\left|\rho_\circ^*\right|^2 e^{-\pi/N_r\omega^*}\right)|e_\rho|^2 \\ &\quad + \frac{N_r\pi}{2\omega^*}\left|\rho_\circ^*\right|^3 |e_\rho|\,|e_\omega|\,. \end{aligned} \tag{57}$$

The last term satisfies

$$\frac{N_r^2\pi}{2\omega^*}\left|\rho_\circ^*\right|^3 |e_\rho|\,|e_\omega| \leq \frac{N_r^2\pi}{4\omega^*}\left|\rho_\circ^*\right|\Big(|e_\rho|^2 + \left|\rho_\circ^*\right|^4 |e_\omega|^2\Big).$$

Then, recalling that $k_{di} = \lambda_i k_d$ and $k_{pi} = \lambda_i k_p$, let

$$k_{d1} \geq N_r^2 \left|\rho_\circ^*\right|^2 \left(3\varepsilon_1 + \frac{\pi}{4\omega^*}\left|\rho_\circ^*\right|^3\right) \tag{58}$$

$$k_{p1} = 1/\varepsilon_1 \tag{59}$$

and

$$\delta_1 := \frac{1}{2}\begin{bmatrix} e_\omega \\ \rho^{*\top}\mathbb{J}^\top e_\rho \end{bmatrix}\begin{bmatrix} k_{d2} & \varepsilon_1 k_d \\ \varepsilon_1 k_d & \varepsilon_1 k_{p2} \end{bmatrix}\begin{bmatrix} e_\omega \\ \rho^{*\top}\mathbb{J}^\top e_\rho \end{bmatrix}$$

which is non-negative if

$$\frac{k_p\lambda_2^2}{k_d} \geq \varepsilon_1\,. \tag{60}$$

Under these conditions, putting together the expressions (22), (53) and (57) we obtain (26).